IMPACT OF A BODY ON A THIN PLATE LYING ON THE SURFACE OF A COMPRESSIBLE LIQUID

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The problem of impact of a body on the surface of a compressible fluid was solved in 1947 by Galin [1]. Later, the analogous problem of the pressure of a punch on an elastic semiplane was considered by Flitman [2].

This paper considers the two-dimensional problem of the impact of a body of finite width on a thin plate lying on the surface of a compressible fluid. The problem is formulated in Section 1. Section 2 indicates the solution of the integro-differential equation for a certain initial time interval; the integral member of the equation has a difference kernel with semi-infinite space change of variables. The solution of this equation is given as a quadrature. Section 3 gives the existence proofs of the solution. Section 4 gives the solution for any instant of time. Section 5 gives simple formulas for the deformation of the plate for small times in the case where the body moves with constant velocity after the impact. Section 6 gives the law of motion of the body when the initial velocity is given and the body moves with variable velocity after the impact. In both cases the forces which act on the body from the fluid and the plate, are determined.

The solution is compared with the solution of the problem of impact of a body on the surface of a compressible fluid with no cover [1]. The effect of the cover on the impact of the body on the fluid is investigated.

The notation is: ρ_0, density of the fluid at rest; c, velocity of sound in the fluid at rest; ρ, density of the plate material; h, plate thickness; l, width of body; $v(t)$, velocity of the moving body; $P(x,y,t)$, excess pressure in the fluid; $w(x,t)$, plate displacement;

$$x_1 = x / l, \quad y_1 = y / l, \quad t_1 = ct / l, \quad w_1 = w / l, \quad P_1 = P / \rho_0 c^2, \quad v_1 = v / c$$

are dimensionless quantities (with the index 1 omitted by consent).

1. Let a thin plate lying on the surface of a compressible fluid at $y = 0$ be struck at time $t = 0$ by a solid symmetrical body (Fig.1) having a flat bottom of width l, and moving initially with a certain velocity $v_y = v(t)$.

In impact problems the solution for an initial time interval $t \sim 1$ is of the greatest interest. Therefore in this problem we shall solve for the time $0 \leqslant t \leqslant 1$.

The solution for any instant of time appears to be quite complex. It will be also stated in a final form without investigation.

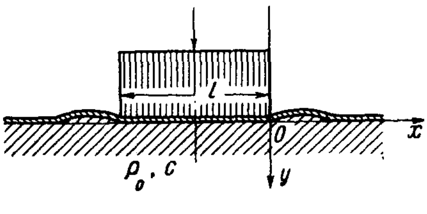

Fig. 1

A shock wave will be propagated in the fluid after the impact. The plate will be deformed under the action of the body and the shock wave. We shall solve the problem on the assumption that the plate does not tear at the edges of the body $x = 0$ and $x = -1$.

As a first approximation to the plate deformation we make the assumption given in [3]. This approximation postulates the existence of an elastic limit to Hooke's law; beyond that limit the plate behaves as a membrane under a constant tension determined by the value of the flow limit σ for tension in the plastic range.

Thus, we shall consider that part of the effect of the shock wave on the plate that takes place in the plastic state, and that its displacement $w(x,t)$ satisfies the membrane equation

$$\frac{\partial^2 w}{\partial t^2} - \alpha^2 \frac{\partial^2 w}{\partial x^2} = -\,\varepsilon p\,(x,\ t) \qquad \left(\alpha = \frac{1}{c}\left[\frac{\sigma}{\rho}\right]^{1/2},\ \ \varepsilon = \frac{\rho_0 l}{\rho h}\right) \tag{1.1}$$

where $p(x,t)$ is the pressure on the plate on the fluid side, and α and ε are dimensionless parameters. In actual cases $\alpha < 1$.

The pressure $P(x,y,t)$ in the compressible fluid filling the semiexpance $y > 0$ is expressed in terms of the normal velocity of motion of the boundary $v_y(x,t)$ by means of the wave potential [4]

$$P\ (x,\ y,\ t) = \frac{1}{\pi}\ \frac{\partial}{\partial t} \int\limits_0^{t-y} d\tau \int\limits_{x_-}^{x_+} \frac{v_y\,(\xi,\ \tau)\, d\xi}{\sqrt{(t-\tau)^2 - (x-\xi)^2 - y^2}}$$

$$(y \geqslant 0,\quad x_\pm = x \pm \sqrt{(t-\tau)^2 - y^2})$$

From this the hydrodynamic pressure on the plate is equal to

$$p\,(x,\ t) = P\ (x,\ 0,\ t) = \frac{1}{\pi}\,\frac{\partial}{\partial t}\int\limits_0^t d\tau \int\limits_{x-(t-\tau)}^{x+(t-\tau)} \frac{v_y\,(\xi,\ \tau)\, d\xi}{\sqrt{(t-\tau)^2 - (x-\xi)^2}} \tag{1.2}$$

for

$$v_y = \begin{cases} \partial w / \partial t & (x < -1,\ x > 0) \\ v\,(t) & (-1 \leqslant x \leqslant 0) \end{cases} \tag{1.3}$$

Up until the shock

$$w = 0, \qquad \partial w / \partial t = 0 \qquad (t \leqslant 0) \tag{1.4}$$

By virtue of symmetry in the problem about the line $x = -\frac{1}{2}$ we have the relation

$$w\ (-\,x,\ t) = w\ (x - 1,\ t) \qquad (x \geqslant 1)$$

and so the function $w(x,t)$ is sufficiently determined for $x > 0$.

We note that in the time interval $0 \leqslant t \leqslant 1$ the left edge of the body at $x = -1$ will not affect the deformation of the plate at $x > 0$. Hence the displacement $w(x,t)$ for $x > 0$ will be the same as if the body's width were semi-infinite $-\infty < x \leqslant 0$.

The condition

$$w(0,\ t) = \int_0^t v(\tau)\, d\tau$$

must be satisfied at the edge of the plate $x = 0$.

We set, therefore,

$$w(x,\ t) = \int_0^{t-x/\alpha} v(\tau)\, d\tau + w_1(x,\ t) \qquad (x > 0) \tag{1.5}$$

The first term on (1.5) corresponds to the solution of the problem of impact of a body on a plate in vacuum, and the second term $w_1(x,t)$ gives the effect of the fluid on the plate deformation.

By recalling (1.1) to (1.5) in connection with the previous remarks, we are led to the problem

$$\frac{\partial^2 w_1}{\partial t^2} - \alpha^2 \frac{\partial^2 w_1}{\partial x^2} + \varepsilon p = 0 \qquad (x > 0) \tag{1.6}$$

$$w_1 = 0, \qquad \frac{\partial w_1}{\partial t} = 0 \qquad (t = 0) \tag{1.7}$$

$$w_1(0,\ t) \equiv 0 \tag{1.8}$$

for the determination of the function $w_1(x,t)$.

Here the function $p(x,t)$ is determined by Formula (1.2), where one must set

$$v_y = \begin{cases} \partial w_1 / \partial t + v(t - x/\alpha) & (x > 0) \\ v(t) & (x < 0) \end{cases} \tag{1.9}$$

2. We multiply both sides of Equations (1.6) by $e^{-\lambda t}$, where $\operatorname{Re}\lambda > \gamma_0$, the exponent of growth of the function $v(t)$, and integrate with respect to t from zero to infinity. As a result, after taking (1.2), (1.7) and (1.9) into account, we get Equation

$$w_1^*(x,\ \lambda) - \frac{\alpha^2}{\lambda^2} \frac{\partial w_1^*(x,\ \lambda)}{\partial x^2} + \frac{\varepsilon}{\pi} \int_0^\infty K_0(\lambda |x - \xi|)\, w_1^*(\xi,\ \lambda)\, d\xi +$$

$$+ \varepsilon \frac{v^*(\lambda)}{\lambda\pi} \int_0^\infty \{K_0(\lambda |x - \xi|)\, e^{-\lambda\xi/\alpha} + K_0[\lambda(x + \xi)]\}\, d\xi = 0 \qquad (x > 0) \tag{2.1}$$

Here K_0 is a MacDonald function; $w_1^*(x,\lambda)$ and $v^*(\lambda)$ are Laplace transforms for the functions $w_1(x,t)$ and $v(t)$, respectively.

The integral equation with a kernel depending on the absolute values of the difference in arguments, with semi-infinite space change of variables, is solved by the Wiener-Hopf-Fok method [2 and 5].

By application of this method to the integro-differential equation (2.1) we get the solution in the form

$$w_1^*(x,\ \lambda) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \lambda W(z,\ \lambda)\, e^{\lambda x z}\, dz \qquad (\gamma > 0) \tag{2.2}$$

Here the function $W(z,\lambda)$ under the integral sign is equal to

$$W(z, \lambda) = -\varepsilon_1 \frac{v^*(\lambda)}{\lambda^2} \frac{\varphi(z, \varepsilon_1)}{1+\alpha z} \frac{1}{\pi} \int_1^\infty \frac{\varphi(\xi, \varepsilon_1)\, d\xi}{(z+\xi)(1-\alpha^2\xi^2)\,\xi\sqrt{\xi^2-1}} \qquad (2.3)$$

where the integral must be taken in the sense of a principal value, and the function $\varphi(z,\varepsilon_1)$ has the form

$$\varphi(z, \varepsilon_1) = \exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \ln\left[1+\frac{\varepsilon_1}{(1-\alpha^2\zeta^2)\sqrt{1-\zeta^2}}\right]\frac{d\zeta}{\zeta-z}\right\} \qquad (2.4)$$

The functions $W(z,\lambda)$ and $\varphi(z,\varepsilon_1)$ are regular in the semiplane $\operatorname{Re} z > 0$; for this $W \to 0$ and $\varphi \to 1$ as $z \to \infty$.

Knowing the function $w_1^*(x,t)$, the function $w_1(x,\lambda)$ is found by an inverse transformation in λ

$$w_1(x, t) = \frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} w_1^*(x, \lambda)\, e^{\lambda t} d\lambda \quad (\gamma_1 > \gamma_0) \qquad (2.5)$$

We take the procedute of obtaining the function $W(z,\lambda)$ with simplified expressions for $\varphi(z,\varepsilon_1)$ and $w^*(x,\lambda)$, as well as of studying the solution $w_1(x,t)$.

Upon multiplying both sides of Equation (2.1) by e^{-kx}, where $\operatorname{Re} k > 0$, and integrating from $x = 0$ to $x = \infty$, we get

$$\left(1-\alpha^2 z^2+\frac{\varepsilon_1}{\sqrt{1-z^2}}\right) W(z, \lambda)+\frac{\varepsilon_1}{\pi}\int_1^\infty\left[W(\xi, \lambda)+\frac{\alpha}{1+\alpha\xi}v^*(\lambda)\right]\times$$

$$\times\frac{d\xi}{(z-\xi)\sqrt{\xi^2-1}}+\varepsilon_1\frac{v^*(\lambda)}{\lambda^2}\left[\frac{\alpha}{(1+\alpha z)\sqrt{1-z^2}}+\frac{1}{\pi}\int_1^\infty\frac{d\xi}{(z+\xi)\,\xi\sqrt{\xi^2-1}}\right]+$$

$$+\frac{\alpha^2}{\lambda^2}v^*(\lambda)+\frac{\alpha^2}{\lambda^2}\left[\frac{\partial w^*(x,\lambda)}{\partial x}\right]_{x=0}=0 \qquad (0<\operatorname{Re} z<1) \qquad (2.6)$$

Here $z = k/\lambda$, $\varepsilon_1 = \varepsilon/\lambda$, and the function

$$W(z, \lambda) = \int_0^\infty e^{-kx} w_1^*(x, \lambda)\, dx \qquad (z = k/\lambda)$$

is a Laplace transform for the unknown function $w_1^*(x,\lambda)$.

We agree to choose for the function $w = \sqrt{1-z^2}$ that branch which is determined in the z-plane without the branches $(-1, -\infty)$ and $(1, +\infty)$ and which takes a positive value for $-1 < z < 1$.

Omitting the reasoning and the transformations required for the proposed method, we get the function $W(z,\lambda)$ from (2.6) in the form (2.3).

The inversion theorem applied to (2.3) gives $w^*(x,\lambda)$ in the form (2.2).

The function $\varphi(z,\varepsilon_1)$, determined in accordance with (2.4), may be presented in the form

$$\varphi(z, \varepsilon_1) = \frac{(1+\alpha z)\sqrt{1+z}}{\alpha(z+a_1)(z+a_2)}\psi(z, \varepsilon_1) \qquad (2.7)$$

where $\alpha_{1,2}$ are roots of the equation $\varepsilon_1 + (1-\alpha^2 z^2)\sqrt{1-z^2} = 0$, lying in the semiplane $\operatorname{Re} z < 0$ under the condition $\operatorname{Re} \varepsilon_1 > 0$ ($\operatorname{Re} \lambda > 0$). For this $a_{1,2} = \sqrt{1-\omega_{1,2}^2}$, and $\omega_{1,2} = {}^1/_2(\Omega_1-\Omega_2) \pm {}^1/_2\, i\sqrt{3}(\Omega_1+\Omega_2)$

where
$$\Omega_{1,2} = \left\{\left[\frac{\varepsilon_1^2}{4\alpha^2} + \left(\frac{1-\alpha^2}{3\alpha^2}\right)^3 \pm \frac{\varepsilon_1}{2\alpha^2}\right]^{1/2}\right\}^{1/3} \qquad (\Omega_{1,2} > 0 \quad \text{for} \quad \varepsilon_1 > 0)$$

The function
$$\psi(z, \varepsilon_1) = \exp\left[\frac{z}{\pi}\int_0^\infty \ln\frac{(\sqrt{1+\xi^2}+\omega_1)(\sqrt{1+\xi^2}+\omega_2)}{\sqrt{1+\xi^2}+\omega_3}\right]\frac{d\xi}{\xi^2+z^2}$$
$$(\omega_3 = \Omega_1 - \Omega_2, \ \operatorname{Re}\omega_3 > 0) \tag{2.8}$$

is regular in the semiplane $\operatorname{Re} z > 0$, behaves as $z^{\frac{1}{2}}$ at infinity, has a branch point at $z = -1$, has no poles in the finite part of the plane, and also has no zeros under the condition $\operatorname{Re}\varepsilon_1 > 0$, and for $\varepsilon_1 = 0$
$$\psi(z, 0) = \frac{1+\alpha z}{\alpha\sqrt{1+z}}$$

If in (2.3) $\alpha \to 0$ $(\sigma \to 0)$ and $\varepsilon_1 \to \infty$ $(h \to 0)$ and it is considered that
$$\lim \sqrt{\varepsilon_1}\,\varphi(z, \varepsilon_1) = (1+\alpha z)\sqrt{1+z} \qquad \text{for } \varepsilon_1 \to \infty$$
then we obtain a function $W(z,\lambda)$ which coincides with the corresponding function found in [2] for a solution to the problem of the pressure of a semi-punch on a compressible fluid, which is to be expected.

It may be established by considering (2.3), (2.4) and (2.7) that the function under the integral sign in (2.2) has one branch point at $z = -1$ in the semiplane $\operatorname{Re} z < 0$ and three poles at the points $z = -a_1$, $z = -a_2$ and $z = -\xi$. The function under the integral sign in the expression for $W(z,\lambda)$ (2.3), has the last pole. Since $\xi \geqslant 1$, the pole $z = -\xi$ lies on the line of the cut $(-1, -\infty)$.

By considering these singularities and by deforming the integration contour in (2.2) we get after the usual transformations
$$w_1^*(x,\lambda) = -\varepsilon\frac{v^*(\lambda)}{\lambda^2}\left\{\frac{2}{\alpha}\frac{\omega_1+\omega_2}{a_1-a_2}\sum_{n=1}^{2}(-1)^n\frac{\sqrt{1-a_n}\,\omega_n f(a_n,\varepsilon_1)}{(\omega_n+\omega_3)\,\psi(a_n,\xi_1)}e^{-\lambda a_n x} + \right.$$
$$\left. + \frac{1}{\pi}\int_0^\infty\left[\frac{1}{\xi} - f(\xi,\varepsilon_1)\right]\frac{(1+\alpha\xi)\sqrt{\xi^2-1}}{\varepsilon_1^2+(1-\alpha^2\xi^2)^2(\xi^2-1)}e^{-\lambda x\xi}\,d\xi\right\} \qquad (\operatorname{Re}\lambda > \gamma_0) \tag{2.9}$$
where
$$f(z,\varepsilon_1) = \frac{1}{\varphi(z,\varepsilon_1)}\frac{1}{\pi}\int_1^\infty\frac{\varphi(\xi,\varepsilon_1)\,d\xi}{(\xi-z)\ (1-\alpha^2\xi^2)\,\xi\sqrt{\xi^2-1}} \tag{2.10}$$

It is seen from (2.10) that the function $f(\xi,\varepsilon_1)$ has a singularity of the form $(\xi-1)^{-1}$ at the point $\xi = 1$. This function enters into (2.9) with a multiplier $(\xi-1)^{1/2}$. It follows that the singularity under the integral sign in (2.9) is integrable at $\xi = 1$.

The function under the integral in (2.10) has a pole at $\xi = 1/\alpha$. This integral must be taken in the sense of a principal value; in this sense it converges.

Noting the dependency on λ of all functions on the right-hand side of (2.9), it is easy to establish that the function $w_1^*(x,\lambda)$ found, is regular in the semiplane $\operatorname{Re}\lambda > \gamma_0$, integrable and twice differentiable with respect to x. For this integration and differentiation the second term in (2.9) may be effected under the integral sign, since the integral up to and after differentiation converges uniformly with λx for $x \geqslant 0$. For this, $w_1^*(\infty,\varepsilon_1) = 0$.

We clarify how the function $w_1^*(x,\lambda)$ behaves for $\operatorname{Im}\lambda \to \pm\infty$. In accordance with (2.9) the function $w_1^*(x,\lambda)$ is continuous for $x \geqslant 0$ when $\operatorname{Re}\lambda = \gamma_1$. We get from (2.9) as $\lambda \to \infty$

$$w_1^*(x,\lambda) = -\varepsilon\frac{v^*(\lambda)}{\lambda^2}\frac{1}{\pi}\int_1^\infty \frac{e^{-\lambda x\xi} - e^{-\lambda x/\alpha}}{(1-d\xi)^2(1+\alpha\xi)\,\xi\sqrt{\xi^2-1}}d\xi + O\left[\frac{v^*(\lambda)}{\lambda^3}\right] \quad (2.11)$$

since for $\varepsilon_1 = 0$ $(\lambda = \infty)$

$$\varphi = 1, \quad \psi = \frac{1+\alpha z}{\alpha\sqrt{1+z}}, \quad a_{1,2} = \frac{1}{\alpha}, \quad \omega_{1,2} = \pm i\left(\frac{1-\alpha^2}{\alpha^2}\right)^{1/2}, \quad \omega_3 = 0$$

$$\frac{\omega_1+\omega_2}{a_1-a_2}\frac{1}{\psi(a_{1,2},0)} = \frac{\alpha}{2}\left(1+\frac{1}{\alpha}\right)^{1/2}$$

It is seen from (2.11) that the function under the integral has a singularity of the form

$$\frac{e^{-\lambda x\xi} - e^{-\lambda x/\alpha}}{(1-\alpha\xi)^2} \approx -\frac{\lambda x}{1-\alpha\xi} \qquad (\lambda \to \infty)$$

This singularity is integrable in the sense of a principal value, and consequently

$$|w_1^*(x,\lambda)| < C\lambda^{-1}v^*(\lambda)$$

where C is a constant.

The function $v^*(\lambda) \to 0$ for $\lambda \to \infty$ (the usual requirement) therefore the integral

$$\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} w_1^*(x,\lambda)\,d\lambda$$

converges absolutely. From this it follows that the function $w_1(x,t)$ determined by the integral (2.5) is continuous for $x \geqslant 0$ and $t \geqslant 0$.

Further, if it is required that the function $v^*(\lambda)$ behave at infinity as

$$v^*(\lambda) \sim \lambda^{-(1+\delta)} \qquad (\lambda \to \infty,\ \delta > 0) \qquad (2.12)$$

then $w_1(x,t)$ must be differentiated once under the integral sign with respect to x and t. After differentiation the integral converges uniformly with x and t and so there exists a continuous first derivative.

Analogously, for condition

$$v^*(\lambda) \sim \lambda^{-(2+\delta)} \qquad (\lambda \to \infty,\ \delta > 0) \qquad (2.13)$$

the function $w_1(x,t)$ may be differentiated twice under the integral sign and so there will exist a continuous second derivative.

3. The solution (2.5) was obtained formally; we establish it briefly on its foundation. The function $W(z,\lambda)$ found, is regular in the semiplane $\operatorname{Re} z > 0$ and reduces to zero as z^{-2} at infinity; therefore it may be represented as an integral of the Cauchy type

$$W(z,\lambda) = -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{W(\zeta,\lambda)}{\zeta - z}d\zeta \qquad (\operatorname{Re} z > 0) \qquad (3.1)$$

By consideration of the properties of the function under the integral in (3.1) and by deforming the contour of integration as with (2.2), we get

$$W(z,\lambda) = -\varepsilon\frac{v^*(\lambda)}{\lambda^2}\left\{\frac{2}{\alpha}\frac{\omega_1+\omega_2}{a_1-a_2}\sum_{n=1}^{2}(-1)^n\frac{\sqrt{1-a_n}\,\omega_n f(a_n,\varepsilon_1)}{(\omega_n+\omega_3)\psi(a_n,\varepsilon_1)}\frac{1}{z+a_n} + \right.$$

$$\left. + \frac{1}{\pi}\int_1^\infty\left[\frac{1}{\xi} - f(\xi,\varepsilon_1)\right]\frac{(1+\alpha\xi)\sqrt{\xi^2-1}}{\varepsilon_1^2+(1-\alpha^2\xi^2)^2(\xi^2-1)}\frac{d\xi}{z+\xi}\right\} (\operatorname{Re} z > 0) \qquad (3.2)$$

It follows from (2.9) and (3.2) that the functions $w_1^*(x,\lambda)$ and $W(z,\lambda)$ are connected by a Laplace transformation.

By assuming that $x = 0$ in (2.2) and recalling that the function $W(z, \lambda)$ is regular in the semiplane $\operatorname{Re} z > 0$ and reduces to zero as z^{-2} at infinity, and after deforming the contour of integration on the semicircle of radius R lying on the right side of the semiplane, we get upon letting $R \to \infty$:

$$w_1^* (0, \lambda) = 0 \tag{3.3}$$

which must hold by virtue of condition (1.8).

We remark that the function $w_1^*(x,\lambda)$ in the form (2.9) will be an exact solution of the integro-differential Equation (2.1).

The integral

$$I_1 (x, \lambda) = \frac{\varepsilon}{\pi} \int_0^\infty K_0 (\lambda | x - \xi |) w_1^* (\xi, \lambda)\, d\xi$$

in Equation (2.1) may be presented in the form

$$I_1 (x, \lambda) = \frac{\varepsilon}{2\pi i} \int_{-i\infty}^{i\infty} \frac{W (z, \lambda)\, e^{\lambda x z}}{\sqrt{1 - z^2}}\, dz \qquad (\operatorname{Re} z > 0) \tag{3.4}$$

if use is made of the integral representation for MacDonald functions [6]

$$K_0 (\lambda | x - \xi |) = \int_0^\infty \frac{\cos [\lambda (x - \xi) \eta]}{\sqrt{1 + \eta^2}}\, d\eta \qquad (\operatorname{Re} \lambda > 0)$$

By transformation of the integral (3.4) like that in (2.2) we get

$$I_1 (x, \lambda) = - \varepsilon \frac{v^* (\lambda)}{\lambda^2} \left\{ \frac{2}{\alpha} \frac{\omega_1 + \omega_2}{a_1 - a_2} \sum_{n=1}^{2} (-1)^n \frac{\varepsilon_1 \omega_n f (a_n, \varepsilon_1)}{\sqrt{1 + a_n} (\omega_n + \omega_3) \psi (a_n, \varepsilon_1)} e^{-\lambda a_n x} + \right.$$
$$+ \frac{1}{\pi} \int_1^\infty \left[\frac{\varepsilon_1^2}{(1 - \alpha\xi)\, \xi \sqrt{\xi^2 - 1}} + (1 - \alpha^2\xi^2)(1 + \alpha\xi) \sqrt{\xi^2 - 1}\, f (\xi, \varepsilon_1) \right] \times$$
$$\left. \times \frac{e^{-\lambda x \xi}\, d\xi}{\varepsilon_1^2 + (1 - \alpha^2\xi^2)^2 (\xi^2 - 1)} \right\} \tag{3.5}$$

Recalling the equality (3.4) and another integral representation of MacDonald functions [6]

$$K_0 [\lambda (x + \xi)] = \int_1^\infty \frac{e^{-\lambda (x+\xi)\eta}}{\sqrt{\eta^2 - 1}}\, d\eta \qquad (x + \xi > 0,\ \operatorname{Re} \lambda > 0)$$

the free term in Equation (2.1) is presented in the form

$$I_1 (x, \lambda) = \varepsilon \frac{v^* (\lambda)}{\lambda \pi} \int_0^\infty \{ K_0 (\lambda | x - \xi |)\, e^{-\lambda \xi / \alpha} + K_0 [\lambda (x + \xi)] \}\, d\xi =$$
$$= \varepsilon \frac{v^* (\lambda)}{\lambda^2 \pi} \int_1^\infty \frac{e^{-\lambda x \xi}}{(1 - \alpha\xi)\, \xi \sqrt{\xi^2 - 1}}\, d\xi \tag{3.6}$$

Upon substitution of (2.9), (3.5) and (3.6) into (2.1), and recalling that $a_{1,2}$ satisfies the algebraic equation $\varepsilon_1 + (1 - \alpha^2 z^2)\sqrt{1 - z^2} = 0$, we get an identity.

It is shown that the function $w_1^*(x,\lambda)$ is actually a solution of the integro-differential equation (2.1).

As has been shown above, the continuous derivatives

$$\frac{\partial^2 w_1}{\partial t^2} = \frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \lambda^2 w_1^* (x, \lambda)\, e^{\lambda t}\, d\lambda, \qquad \frac{\partial^2 w_1}{\partial x^2} = \frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{\partial^2 w_1^* (x, \lambda)}{\partial x^2} e^{\lambda t}\, d\lambda \tag{3.7}$$

exist under conditions (2.13).

The function

$$p(x, t) = \frac{1}{2\pi i} \int\limits_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \lambda^2 [I_1(x, \lambda) + I_2(x, \lambda)] e^{\lambda t} d\lambda \tag{3.8}$$

will be continuous, evidently, even for the weaker condition (2.12).

By substitution of (3.7) and (3.8) inot (1.6), and after recalling (2.1), we get an identity.

By assuming $t = 0$ in (2.5) and in the first derivative with respect to t of the integral of (2.5), and after deformation of the contour of integration as in (3.3), we get $w_1 = 0$, $\partial w_1/\partial t = 0$ for $t = 0$. It follows from (3.3) that $w_1(0, t) = 0$.

The solution $w_1(x, t)$ in the form (2.5) under conditions (2.13) will be an exact solution of Equation (1.6) fir conditions (1.7) and (1.8).

The uniqueness of the solution is evident.

In the case where the function $v(t)$ satisfies condition (2.12), the derivatives (3.7) will not be continuous. Nevertheless the combination of the second derivatives

$$\frac{\partial^2 w_1}{\partial t^2} - \alpha^2 \frac{\partial^2 w_1}{\partial x^2} = \frac{1}{2\pi i} \int\limits_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \left[[\lambda^2 w^*(x, \lambda) - \alpha^2 \frac{\partial^2 w_1^*(x, \lambda)}{\partial x} \right] e^{\lambda t} d\lambda$$

will be continuous, as follows from (2.11).

We examine the discontinuities in the second derivatives. For this we take the function $v(t)$ in the form

$$v(t) = Vt^{\delta} \qquad (V = \text{const}, \ \delta > 0) \tag{3.9}$$

Then

$$v^*(\lambda) = V\Gamma(1 + \delta)\lambda^{-(1+\delta)} \tag{3.10}$$

where Γ is a gamma function. By substitution of (3.10) into (2.5) and recalling Equation (2.11) for small t, we get

$$w_1 = \frac{-\varepsilon V}{(1 + \delta)(2 + \sigma)\pi} \int\limits_1^{\infty} \frac{h(t - x\xi)(t - x\xi)^{2+\delta} - h(t - x/\alpha)(t - x/\alpha)^{2+\delta}}{(1 - \alpha\xi)^2 (1 + \alpha\xi) \xi \sqrt{\xi^2 - 1}} d\xi + O(t^3) \tag{3.11}$$

Here $h(t) = 1$ for $t > 0$ and $h(t) = 0$ for $t < 0$.

It follows from (3.11) that the second derivatives with respect to x and t suffer discontinuities of the second kind on the line $x = \alpha t$, a bending wave front propagated over the plate with velocity $a = \sqrt{\sigma/\rho}$. This is to be expected, since according to (1.5)

$$w(x, t) = V(1 + \delta)^{-1}(t - x/\alpha)^{1+\delta} + w_1(x, t)$$

the second derivatives suffer discontinuity at $x = 0$, $t = 0$. Since the differential operator in (1.1) is a wave operator, the discontinuity will be conserved for $t > 0$ in $w_1(x, t)$, demonstrating the effect of the fluid on the plate deformation, and the discontinuity will be propagated in the plate with a dimensionless velocity α.

4. The solution (2.11) is found for the time $0 \leqslant t \leqslant 1$. For the time $0 \leqslant t \leqslant \infty$ the problem reduces to the solution of the integro-differential equation, more complicated than (2.1). It has the form

$$w_1^*(x,\lambda) - \frac{\alpha^2}{\lambda^2}\frac{\partial^2 w^*(x,\lambda)}{\partial x^2} + \frac{\varepsilon}{\pi}\int_0^\infty \{K_0(\lambda\,|x-\xi|) + K_0[\lambda(x+1+\xi)]\}\times$$

$$\times w_1^*(\xi,\lambda)\,d\xi + \varepsilon\frac{v^*(\lambda)}{\lambda\pi}\left\{\int_0^\infty [K_0(\lambda\,|x-\xi|) + K_0(\lambda(x+1+\xi))]\,e^{-\lambda\xi/\alpha}\,d\xi + \right.$$

$$\left. + \int_0^1 K_0[\lambda(x+\xi)]\,d\xi\right\} = 0 \qquad (x \geqslant 0) \qquad (4.1)$$

Application of the Wiener-Hopf-Fok method to Equation (4.1) leads to a Fredholm equation of the second kind for the function $W(z,\lambda)$

$$W(z,\lambda) + \frac{\varepsilon_1\varphi(z,\varepsilon_1)}{(1+\alpha z)\pi}\int_1^\infty \frac{e^{-\lambda\xi}\varphi(\xi,\varepsilon_1)\,W(\xi,\lambda)}{(z+\xi)(1+\alpha\xi)\sqrt{\xi^2-1}}\,d\xi = -\frac{\varepsilon_1 v^*(\lambda)}{\lambda^2}\frac{\varphi(z,\varepsilon_1)}{1+\alpha z}\Phi(z,\lambda)$$

$$(\operatorname{Re} z > 0) \qquad (4.2)$$

By solving Equation (4.2) by the method of successive approximations, we get

$$W(z,\lambda) = -\varepsilon_1\frac{v^*(\lambda)}{\lambda^2}\frac{\varphi(z,\varepsilon_1)}{1+\alpha z}\Big[\Phi(z,\lambda) +$$

$$+ \sum_{n=1}^\infty\left(-\frac{\varepsilon_1}{\pi}\right)^n\int_1^\infty\cdots\int_1^\infty \Phi(\xi_n,\lambda)\prod_{i-1}^n \frac{\varphi^2(\xi_i,\varepsilon_1)\,e^{-\lambda\xi_i}d\xi_i}{(\xi_{i-1}+\xi_i)(1+\alpha\xi_i)^2\sqrt{\xi_i^2-1}}\Big] \qquad (\xi_0 = z) \quad (4.3)$$

Here the function

$$\Phi(z,\lambda) = \frac{1}{\pi}\int_1^\infty \frac{(1-e^{-\lambda\xi})\,\varphi(\xi,\varepsilon_1)}{(z+\xi)(1-\alpha^2\xi^2)\,\xi\sqrt{\xi^2-1}}\,d\xi$$

As a result, we find the function $w_1(x,t)$ to be of the form

$$w_1(x,t) = \frac{1}{(2\pi i)^2}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} e^{\lambda t}\,d\lambda\int_{\gamma-i\infty}^{\gamma+i\infty}\lambda W(z,\lambda)\,e^{\lambda xz}\,dz \qquad \begin{pmatrix}\gamma_1>\gamma_0, & \gamma>0;\\ x>0, & 0\leqslant t<\infty\end{pmatrix} \qquad (4.4)$$

It may be shown that the number of terms in (4.3) will not be infinite as in (4.2), but will be finite over a small dimensionless time t.

5. We consider the case where the body after impact begins to move with constant velocity $v(t) = V$.

The solution may evidently be obtained as a limit of the general solution corresponding to the case in (3.9), for $\delta \to 0$.

By recalling (3.11) and (3.12), letting $\delta \to 0$ in (3.11), and integrating with respect to ξ, we get for small times with an accuracy to t^2

$$w = 0 \quad \text{for } \eta \geqslant 1, \qquad w = -\varepsilon V t^2 f(\eta;\alpha) \quad \text{for } \alpha \leqslant \eta \leqslant 1 \qquad (5.1)$$

$$w = V\{t(1-\alpha^{-1}\eta) + \varepsilon t^2[(1-\alpha^{-1}\eta)^2 f(0;\alpha) - f(\eta;\alpha)]\} \quad \text{for } 0 \leqslant \eta \leqslant \alpha$$

Here

$$\eta = \frac{x}{t}, \qquad f(\eta;\alpha) \quad \frac{1}{2\pi}\left\{\cos^{-1}\eta - \frac{(\eta+\alpha)^2}{4\alpha\sqrt{1-\alpha^2}}\ln\frac{1+\alpha\eta+\sqrt{(1-\alpha^2)(1-\eta^2)}}{\eta+\alpha} - \right.$$

$$-\left[\frac{(\eta-\alpha)(\eta+3\alpha)}{4\alpha\sqrt{1-\alpha^2}} - \frac{(\eta-\alpha)^3}{2\alpha(\sqrt{1-\alpha^2})^3}\right]\ln\left|\frac{1-\alpha\eta+\sqrt{(1-\alpha^2)(1-\eta^2)}}{\eta-\alpha}\right| -$$

$$\left. - \frac{(\eta-\alpha)\sqrt{1-\eta^2}}{2(1-\alpha^2)}\right\} \qquad (5.2)$$

Upon letting $\alpha \to 1$ (5.2) we get

$$f(\eta;1) = {}^1/_2\,\pi^{-1}\left[\cos^{-1}\eta + {}^3/_{16}(1-\eta)\sqrt{1-\eta^2}\right]$$

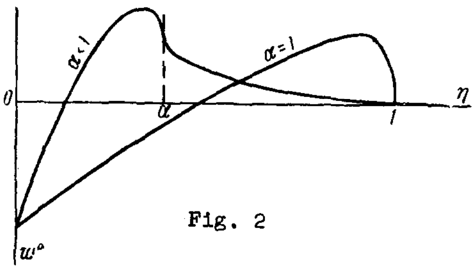

Fig. 2

Fig.2 shows the form of plate deformation $w^\circ = w^\circ(\eta)$ from Formula (5.1); for this $w^\circ = w/Vt$. The solution is continuous in this case and the derivatives suffer discontinuity on the line $x = \alpha t$. The curve of w° has a vertical tangent at the point $\eta = \alpha$ (Fig.2).

We determine the force $F(t)$ acting on the body from the fluid and plate. Evidently

$$F(t) = \int_{-1}^{0} p(x,t)\,dx, \qquad F^*(\lambda) = \int_{-1}^{0} p^*(x,\lambda)\,dx \tag{5.3}$$

It follows from (1.2) that

$$p^*(x,\lambda) = \frac{\lambda V}{\pi}\int_{-1}^{0} K_0(\lambda|x-\xi|)\,d\xi + \frac{\lambda^2}{\pi}\int_{-\infty}^{-1} K_0(\lambda|x-\xi|)\,w^*(\xi,\lambda)\,d\xi +$$

$$+\frac{\lambda^2}{\pi}\int_{0}^{\infty} K_0(\lambda|x-\xi|)\,w^*(\xi,\lambda)\,d\xi = \frac{V}{\lambda\pi}\int_{1}^{\infty}\frac{2-e^{\lambda x\xi}-e^{-\lambda(1+x)\xi}}{\xi\sqrt{\xi^2-1}}\,d\xi +$$

$$+\frac{\lambda^2}{\pi}\int_{0}^{\infty}\{K_0[\lambda(x+\xi)] + K_0[\lambda(x+1+\xi)]\}\,w^*(\xi,\lambda)\,d\xi \qquad (-1\leqslant x\leqslant 0) \tag{5.4}$$

In the case considered the function

$$w^*(x,\lambda) = V\lambda^{-2}e^{-\lambda x/\alpha} + w_1^*(x,\lambda) \tag{5.5}$$

By substitution of (5.5) into (5.4), then (5.4) into (5.3), and integrating with respect to x, we get

$$F^*(\lambda) = \frac{V}{\lambda} - \frac{2V}{\lambda^2\pi}\int_{1}^{\infty}\frac{1-e^{-\lambda\xi}}{(1+\alpha\xi)\,\xi^2\sqrt{\xi^2-1}}\,d\xi + \frac{2\lambda}{\pi}\int_{1}^{\infty}\frac{1-e^{-\lambda\xi}}{\xi\sqrt{\xi^2-1}}\,W(\xi,\lambda)\,d\xi \tag{5.6}$$

The function $W(\xi,\lambda)$ entering into (5.6) has a rather complicated form (2.3). For the inverse transformation of (5.6) we use the first expansion theorem for Laplace transformations [7]. According to this theorem if the transform is expanded in a Laurent series with $\lambda \to \infty$, then the original series is in powers of t which converge uniformly and which represent in themselves the complete function.

For $\lambda \to \infty$ we have $\varepsilon_1 \to 0$. We denote

$$\chi(x\ \varepsilon_1) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\ln\left[1+\frac{\varepsilon_1}{(1-\alpha^2\zeta^2)\sqrt{1-\zeta^2}}\right]\frac{d\zeta}{x-\zeta}$$

$$(x \geqslant 1) \tag{5.7}$$

and in agreement with (2.4) we shall have

$$\varphi(x,\varepsilon_1) = e^{\chi} = \sum_{n=0}^{\infty}\frac{\chi^n(x,\varepsilon_1)}{n!} \tag{5.8}$$

The series (5.8) converges uniformly with x for $x \geqslant 1$ and for small

$|\varepsilon_1|$, since for $\varepsilon_1 \to 0$ the function $\chi \to 0$.

For small $|\varepsilon_1|$ the function is expanded in a uniformly converging series

$$\chi(x, \varepsilon_1) = \sum_{m=1}^{\infty} \frac{\chi_m(x)}{m} \varepsilon_1^m \tag{5.9}$$

Here

$$\chi_m(x) = \frac{(-1)^m}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\zeta}{(1-\alpha^2\zeta^2)^m (1-\zeta^2)^{m/2} (x-\zeta)} =$$

$$= \frac{(-1)^m}{\pi} \int_0^{\infty} \frac{x\,d\xi}{(x^2+\xi^2)(1+\alpha^2\xi^2)^m (1+\xi^2)^{m/2}} \tag{5.10}$$

By substitution of (5.9) into (5.8) we shall have

$$\varphi(x, \varepsilon_1) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{m=1}^{\infty} \frac{\chi_m}{m} \varepsilon_1^m \right]^n =$$

$$= 1 + \sum_{n=1}^{\infty} \varepsilon_1^n \sum_{m=1}^{n} \frac{1}{m!} \sum_{(q_1, \ldots, q_k)} \left(\frac{\chi_1}{1}\right)^{q_1} \cdots \left(\frac{\chi_k}{k}\right)^{q_k}$$

Here the summation is taken over those combinations of whole numbers $q_1, q_2, \ldots, q_k$ equal to 1,2,.... for which

$$\sum_{i=1}^{k} q_i = m, \qquad \sum_{i=1}^{k} i q_i = n$$

We get after analogous transformations

$$\varphi(z, \varepsilon_1)\varphi(\xi, \varepsilon_1) = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{\lambda^n} \varphi_n(z, \xi) \qquad \left(\varepsilon_1 = \frac{\varepsilon}{\lambda}\right) \tag{5.11}$$

where

$$\varphi_n(z, \xi) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{(q_1, \ldots, q_k)} \left[\frac{\chi_1(z) + \chi_1(\xi)}{1}\right]^{q_1} \cdots \left[\frac{\chi_k(z) + \chi_k(\xi)}{k}\right]^{q_k}$$

By recalling (2.3) and (5.12) and carrying out the inverse transformation of (5.6) we obtain as a result the force $F(t)$ in the form

$$F(t) = V\left[1 - \sum_{n=1}^{\infty} \varepsilon^{n-1} C_n(\alpha) t^n\right] \tag{5.12}$$

where the coefficients $C_n(\alpha)$ are

$$C_1 = \frac{2}{\pi}\left(1 - \frac{\alpha\pi}{2} - \frac{\alpha^2}{\sqrt{1-\alpha^2}} \ln \frac{1+\sqrt{1-\alpha^2}}{\alpha}\right)$$

$$C_2 = \frac{1}{2}\left(1 + \frac{2\alpha^2}{\sqrt{1+\alpha^2}} \ln \frac{1+\sqrt{1-\alpha^2}}{\alpha}\right) C_1(\alpha) + \frac{1}{\pi} \int_1^{\infty} \frac{\ln(\xi + \sqrt{\xi^2-1})\,d\xi}{\xi^2(1+\alpha\xi)^2(\xi^2-1)(1-\alpha\xi}$$

$$C_n = \frac{2}{(n+2)\pi^2} \int_1^{\infty} \frac{d\xi}{(1+\alpha\xi)\xi\sqrt{\xi^2-1}} \int_1^{\infty} \frac{\varphi_n(\xi, \eta)\,d\eta}{(\xi+\eta)(1-\alpha^2\eta^2)\eta\sqrt{\eta^2-1}} \quad (n = 3, 4, \ldots)$$

It may be shown that the series (5.12) converges rapidly.

In case the plate is absent $(\varepsilon = \infty,\ \alpha = 0)$ a simple expression for the force $F_0(t)$ acting on the body from the fluid, from (2.3) and (5.6), as well as from the results of [1]

$$F_0(t) = V(1-t) \qquad (0 \leqslant t \leqslant 1) \tag{5.13}$$

By calculating the coefficients in the series (5.12) by numerical integration for given values of α and ε, an estimate may be made of the effect of the plate on the impact of body and fluid by comparing (5.12) with (5.13).

6. Let the initial velocity of the body at the instant of impact be equal to V_0, and afterwards let the body move with a certain unknown velocity $v(t)$. We determine the law of motion of the body.

The force acting on the body from fluid and plate will be

$$R(t) = \frac{d}{dt}\int_0^t \frac{F(t-\tau)}{V} v(\tau)\, d\tau \tag{6.1}$$

according to the principle of Duhamel. Here the function $F(t)$ must be taken from the solution of the preceding problem in agreement with Formula (5.12).

The force $R(t)$ is directed against motion of the body and therefore its equation of motion in dimensionless variables will have the form

$$\frac{dv}{dt} = -\mu R(t) \qquad \left(\mu = \frac{\rho_0 l^2}{m}\right) \tag{6.2}$$

Here m is the mass of the body per unit length. In real cases the parameter μ is small.

By recalling (5.12) and substituting (6.1) into (6.2) we get the integro-differential equation

$$\frac{dv(t)}{dt} + \mu v(t) - \mu \int_0^t K(t-\tau)\, v(\tau)\, d\tau = 0, \qquad K(t) = \sum_{n=1}^{\infty} n\varepsilon^{n-1} C_n(\alpha)\, t^{n-1} \tag{6.3}$$

Retaining only the first term in the kernel and solving (6.3) by operational methoas for the condition $v(0) = V_0$, we get an approximate solution

$$v(t) = V_0 e^{-0.5\mu t}\,[\cosh(0.5\,\varkappa t) - \varkappa^{-1}\mu \sinh(0.5\,\varkappa t)] \qquad (\varkappa = \sqrt{\mu^2 + 4\mu C_1(\alpha)}) \tag{6.4}$$

which becomes more accurate the smaller is t.

It follows from (6.4) that

$$dv/dt < 0, \qquad d^2v/dt^2 > 0, \qquad dv/d\alpha < 0, \qquad dv/d\mu < 0$$

Knowing $v(t)$, we find the resistance force $R(t)$ from (6.1) as the obvious function

$$R(t) = V_0 e^{-0.5\mu t}\,[\cosh(0.5\,\varkappa t) - \varkappa^{-1}(\mu + 2C_1(\alpha)) \sinh(0.5\,\varkappa t)] \tag{6.5}$$

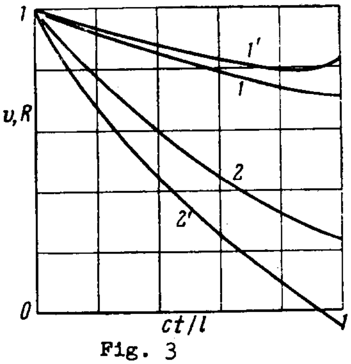

Fig. 3

From (6.5) it follows that

$$dR/dt < 0, \qquad d^2R/dt^2 > 0$$
$$\partial R/\partial\alpha > 0, \qquad \partial R/\partial\mu < 0$$

Results of numerical calculation show that:

1) if m is large ($\mu \approx 0$), then $v \approx V_0$ and R depends strongly on the parameter α;

2) if the parameter μ increases (mass of the body decreases), then the dependence of v on α is strengthened, and that of R on α is diminished.

From this the conclusion follows at once, if we recall that $\alpha = c^{-1}\sqrt{\sigma/\rho}$, that the velocity v of the body diminishes with increasing σ/ρ, and the resistance force R on the body increases.

It is known [1] that in case of impact of a body on the surface of a compressible fluid, its velocity $v_0(t)$ and the force $R_0(t)$ of the fluid on the body, are equal to

$$v_0(t) = V_0 e^{-0.5\mu t}[\cosh(0.5\varkappa_0 t) - \varkappa_0^{-1}\mu\sinh(0.5\varkappa_0 t)] \qquad (6.6)$$

$$(\varkappa_0 = \sqrt{\mu^2 + 4\mu})$$

$$R_0(t) = V_0 e^{-0.5\mu t}[\cosh(0.5\varkappa_0 t) - \varkappa_0^{-1}(\mu + 2)\sinh(0,5\varkappa_0 t)] \qquad (6.7)$$

for $0 \leqslant t \leqslant 1$.

Numerical estimates may be made from Formulas (6.4) to (6.7) of the effect of the plate on the impact of a body on a compressible fluid.

In the example of Fig. 3, curves 1, 2 and 1', 2' have been drawn from Formulas (6.4), (6.5) and (6.6), (6.7), corresponding to $\alpha = 0.8$ and $\mu = 0.4$. As is seen from the curves, the effect of the plate on the velocity of the body is insignificant, but that there is a greater effect on the force of resistance.

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